**DIFFERENCES OF IDEMPOTENTS IN $C^*$-ALGEBRAS**

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Abstract: Suppose that $P$ and $Q$ are idempotents on a Hilbert space $\mathcal{H}$, while $Q = Q^*$ and $I$ is the identity operator in $\mathcal{H}$. If $U = P - Q$ is an isometry then $U = U^*$ is unitary and $Q = I - P$. We establish a double inequality for the infimum and the supremum of $P$ and $Q$ in $\mathcal{H}$ and $P - Q$. Applications of this inequality are obtained to the characterization of a trace and ideal $F$-pseudonorms on a $W^*$-algebra. Let $\varphi$ be a trace on the unital $C^*$-algebra $\mathcal{A}$ and let tripotents $P$ and $Q$ belong to $\mathcal{A}$. If $P - Q$ belongs to the domain of definition of $\varphi$ then $\varphi(P - Q)$ is a real number. The commutativity of some operators is established.

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**Introduction**

Let $P$ and $Q$ be idempotents on a Hilbert space $\mathcal{H}$. Various properties (invertibility, Fredholm property, nuclearity, positivity, etc.) of the difference $P - Q$ were studied in [1–6]. Each tripotent ($A = A^3$) is the difference $P - Q$ of some idempotents $P$ and $Q$ with $PQ = QP = 0$ [7, Proposition 1]. Therefore, tripotents inherit some properties of idempotents [8].

In this article, we obtain some new results on $P - Q$. We prove that the isometry of $U = P - Q$, where $Q^* = Q$, implies the unitarity of $U$ and the equality $Q = I - P$ (Theorem 1). We give an example showing the substantiality of the condition $Q^* = Q$. If $P^* = P$ then $P \wedge Q + P^\perp \wedge Q \leq |P - Q| \leq P \vee Q - P \wedge Q$ with equality in the second inequality if and only if $PQ = QP$ (Theorem 2 and Proposition 1). Using this operator inequality, we establish a new inequality that characterizes traces on a $W^*$-algebra $\mathcal{A}$ (Corollary 4). Applications are obtained to ideal $F$-pseudonorms on $\mathcal{A}$ (Corollary 7).

Let $\varphi$ be a trace on a unital $C^*$-algebra $\mathcal{A}$, let $\mathfrak{A}_\varphi$ be the domain of definition of $\varphi$, and let $P$ and $Q$ belong to $\mathcal{A}$. If $P - Q \in \mathfrak{A}_\varphi$ then $\varphi(P - Q) \in \mathbb{R}$ (Theorem 3). Theorem 3 is a $C^*$-analog of the following familiar assertion [6]: If $P$ and $Q$ are idempotents in $\mathcal{H}$ and $P - Q$ belongs to the ideal $\mathfrak{S}_1$ of trace class operators then the canonical trace $\text{tr}(P - Q)$ belongs to $\mathbb{Z}$. Let $\mathcal{A}$ be a $C^*$-algebra and let $(\mathcal{E}, \preceq)$ be a partially ordered set. We establish a monotonicity criterion for a mapping from $\mathcal{A}^+$ into $\mathcal{E}$ (Proposition 2).

1. **Definitions and Notations**

By a $C^*$-algebra we mean a complex Banach $*$-algebra $\mathcal{A}$ such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$. Given a $C^*$-algebra $\mathcal{A}$, denote by $\mathcal{A}^\text{id}$, $\mathcal{A}^\text{sa}$, and $\mathcal{A}^\text{+}$ the sets of its idempotents, Hermitian elements, and positive elements respectively. If $A \in \mathcal{A}$ then $|A| = \sqrt{A^*A} \in \mathcal{A}^\text{+}$. If $A \in \mathcal{A}^\text{sa}$ then $A_+ = (|A| + A)/2$ and $A_- = (|A| - A)/2$ lie in $\mathcal{A}^\text{+}$ and $A = A_+ - A_-$, $A_+ A_- = 0$. A $W^*$-algebra is a $C^*$-algebra $\mathcal{A}$ having a predual Banach space $\mathcal{A}_\text{sa} : \mathcal{A} \simeq (\mathcal{A}_\text{sa})^*$. For a $W^*$-algebra $\mathcal{A}$, denote by $\mathcal{A}^\text{sa}$ and $\mathcal{A}^\text{sa}$ its subsets of unitary elements and the projection lattice respectively. If $I$ is the unity of $\mathcal{A}$ and $P \in \mathcal{A}^\text{id}$ then $P^\perp = I - P \in \mathcal{A}^\text{id}$.

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