$F(R)$ bigravity

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Preliminary

"bigravity" $=$ system of massive spin 2 field (massive graviton)

$+$ gravity (includes massless spin 2 field $=$ graviton)

$F(R)$ extension of bigravity, Application to Cosmology

Mainly based on

S. Nojiri and S. D. Odintsov,
"Ghost-free $F(R)$ bigravity and accelerating cosmology,"
Phys. Lett. B **716**, 377 (2012) [arXiv:1207.5106 [hep-th]].

S. Nojiri, S. D. Odintsov, and N. Shirai,
"Variety of cosmic acceleration models from massive $F(R)$ bigravity,"
JCAP **1305** (2013) 020

Dark energy

Universe can be regarded as isotropic and homogeneous in the scale larger than the clusters of galaxies
$\Rightarrow$ Friedmann-Robertson-Walker (FRW) metric:

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 \sum_{i,j=1}^{3} \tilde{g}_{ij} dx^i dx^j \, .$$

$a(t)$: scale factor, $\tilde{g}_{ij}$: spacial metric
$\tilde{R}_{ij} = 2K\tilde{g}_{ij}$ ($\tilde{R}_{ij}$: Ricci curvature given by $\tilde{g}_{ij}$)
$K = 1$: unit sphere, $K = -1$: unit hyperboloid, $K = 0$: flat space

$$\begin{cases} da(t)/dt > 0 & \text{: expanding universe} \\ d^2a(t)/dt^2 > 0 & \text{: accelerating expansion} \end{cases}$$

Assume the Universe is filled with perfect fluids.
1st FRW equation: (t, t) component of the Einstein eq.

$$0 = -\frac{3}{\kappa^2}H^2 - \frac{3K}{\kappa^2 a^2} + \rho\,, \quad \kappa^2 \equiv 8\pi G$$

2nd FRW equation: (i, j) component

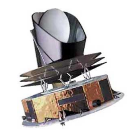
$$0 = \frac{1}{\kappa^2}\left(2\frac{dH}{dt} + 3H^2\right) + \frac{K}{\kappa^2 a^2} + p\,,$$

ρ: energy density, p: pressure, $H \equiv (1/a)\,da(t)/dt$: Hubble rate

The Hubble constant H_0: the present value of H.
$H_0 \sim 70\,\mathrm{km\,s^{-1}Mpc^{-1}} \sim 10^{-33}\,\mathrm{eV}$ in the unit $\hbar = c = 1$.

Cosmic Microwave Background Radiation (CMB) $\Rightarrow K \sim 0$

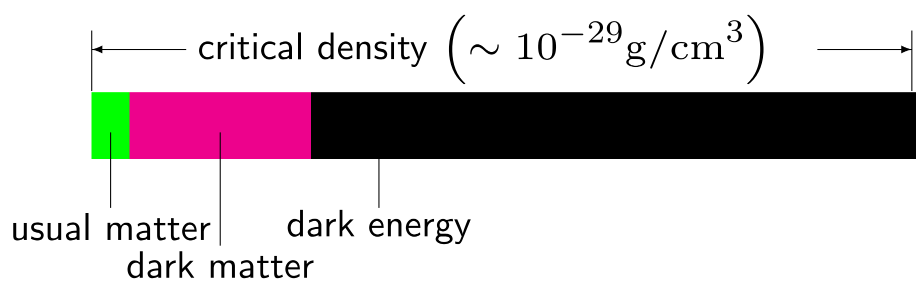
$$\rho \sim \rho_c \equiv \frac{3}{\kappa^2} H_0^2 \sim \left(10^{-3}\,\mathrm{eV}\right)^4 \sim 10^{-29}\mathrm{g/cm}^3\,.$$



Planck satellite

ρ_c: critical density. Flat universe $\Rightarrow \rho \sim \rho_c$

Density of usual matter $\sim 4.9\%$, dark matter $\sim 26.8\%$ of ρ_c
$\Longrightarrow$ something unknown $\sim 68.3\% \cdots$ dark energy



Type Ia Supernovae

$\Rightarrow$ accelerating expansion started about 5 billion years ago.

1st and 2nd FRW equations $\Rightarrow$

$$\frac{1}{a}\frac{d^2a(t)}{dt^2} = \frac{dH}{dt} + H^2 = -\frac{\kappa^2}{6}(\rho + 3p) \ .$$

accelerating expansion $\Rightarrow p < -\rho/3$

Dark energy: large negative pressure

Equation of state (EoS) parameter: $w \equiv \frac{p}{\rho}$

Dark energy: $w \sim -1$

Radiation: $w = 1/3$,

Usual matter, cold dark matter (CDM): $w \sim 0$ (dust),

Cosmological constant: $w = -1$

Dark energy = Cosmological constant??

When EoS parameter w: constant $\Rightarrow$ conservation law:

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0\,,$$

$\Rightarrow \rho = \rho_0 a^{-3(1+w)}$ $(w \neq -1)$, ρ_0: constant of integration

1st FRW eq. $\Rightarrow$

In case $w > -1$, $a(t) \propto t^{\frac{2}{3(1+w)}}$

In case $w < -1$, $a(t) \propto (t_0 - t)^{\frac{2}{3(1+w)}}$

When $t = t_0$, $a(t)$ diverges: Big Rip singularity

In case $w = -1$, $a(t) \propto a_0 \mathrm{e}^{H_0 t}$, $H_0 \equiv \frac{\rho_0 \kappa^2}{3}$, de Sitter space-time

Fine-tuning problem and Coincidence problem

Fine-tuning problem, Coincidence problem:

The definitions slightly depend on persons.

A. 1st and 2nd FRW equations ($K = 0$)

$$0 = -\frac{3}{\kappa^2}H^2 + \frac{\Lambda}{2\kappa^2} + \rho_{\rm matter}\,, \quad 0 = \frac{1}{\kappa^2}\left(2\frac{dH}{dt} + 3H^2\right) - \frac{\Lambda}{2\kappa^2} + p_{\rm matter}\,,$$

Λ: cosmological constant

If the dark energy is the cosmological term, the cosmological constant is unnaturally small.

$$\Lambda \sim \left(10^{-33}\,{\rm eV}\right)^2 \ll M_{\rm Planck} \sim 1/\kappa \sim 10^{19}\,{\rm GeV} = 10^{28}\,{\rm eV}$$

B. Anthropic principle?

$\frac{\Lambda}{2\kappa^2} \sim \rho_{\mathrm{matter}}$ (including dark matter) Very accidental! if Λ is a constant

Age of the Universe: 13.7 billion years

$$\sim \left(10^{-33}\,\mathrm{eV}\right)^{-1} \sim \Lambda^{-\frac{1}{4}}$$

Present temperature of the Universe: $(3K)$

$$\sim 10^{-3}\,\mathrm{eV} \sim (\rho_{\mathrm{matter}})^{1/4} \sim \left(\frac{\Lambda}{2\kappa^2}\right)^{1/4}$$

$\Rightarrow$ Dark energy might be dynamical?

C. Initial condition?
If the dark energy is a perfect fluid whose EoS parameter $w \sim -1$,

$$\rho_{\mathrm{DE}} = \rho_{\mathrm{DE}\,0} a^{-3(1+w)} \sim \rho_{\mathrm{DE}\,0}$$

Usual matter or CDM (dust with $w = 0$)

$$\rho_{\mathrm{matter}} = \rho_{\mathrm{matter}\,0} a^{-3}$$

Ratio of densities of the dark energy to usual matter and dark matter

$$\rho_{\mathrm{DE}}/\rho_{\mathrm{matter}} \sim \left(\rho_{\mathrm{DE}\,0}/\rho_{\mathrm{matter}\,0}\right) a^{-3}$$

In order that $\rho_{\mathrm{DE}\,0} \sim \rho_{\mathrm{matter}\,0}$ in the present Universe,

because the ratio is given by $\rho_{\mathrm{DE}}/\rho_{\mathrm{matter}} \sim a^{-3}$,
When transparent to radiation ($a \sim 10^{-3}$), for example:

$$\rho_{\mathrm{DE}}/\rho_{\mathrm{matter}} \sim 10^{-9}$$

We need to fine-tune the initial condition of the ratio.

There might be a model where the dark matter interacts with dark energy and there is a transition between them?
The EoS parameter of the dark energy changes dynamically depending on the expansion (tracker model)?

D. If the dark energy is the vacuum energy,

the quantum corrections from the matter diverge $\sim \Lambda_{\rm cutoff}^4$.
$\Lambda_{\rm cutoff}$: cutoff scale

If the supersymmetry is restored in the high energy,
the vacuum energy by the quantum corrections $\sim \Lambda_{\rm cutoff}^2 \Lambda_{\cancel{\rm SUSY}}^2$
$\Lambda_{\cancel{\rm SUSY}}$: the scale of the supersymmetry breaking.

If we use the counter term in order to obtain the very small vacuum energy $\left(10^{-3}\,{\rm eV}\right)^4$, we need very very fine-tuning and extremely unnatural.

Maybe we do not understand quantum gravity?

The Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu}$$

What we consider the dark energy as a perfect fluid filling the Universe corresponds to the modification of the energy momentum tensor $T_{\mu\nu}$ of matters, which appears in the r.h.s. in the Einstein equation. On the other hand, there are many models to consider the modification of the Einstein tensor in the l.h.s., which are called modified gravity models.

$F(R)$ gravity, scalar-tensor theory (Brans-Dicke type model), Gauss-Bonnet gravity, $F(G)$ gravity, massive gravity, bigravity...

Recently there have been remarkable progresses in the study of massive gravity and bigravity.

Massive Gravity (theory of massive spin two field)

Fierz-Pauli action (linearized or free theory), 3/4 century ago

M. Fierz and W. Pauli, "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field," Proc. Roy. Soc. Lond. A **173** (1939) 211.

The Lagrangian of the massless spin-two field (graviton) $h_{\mu\nu}$ is given by

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\lambda h^\lambda{}_\mu \partial_\nu h^{\mu\nu} - \partial^\mu h_{\mu\nu}\partial^\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h\,, \quad \left(h \equiv h^\mu{}_\mu\right)\,.$$

Massless graviton: 2 degrees of freedom (helicity),
Massive graviton: 5 degrees of freedom ($2s+1$, spin $s=2$).

The Lagrangian of the massive graviton with mass m is given by

$$\mathcal{L}_m = \mathcal{L}_0 - \frac{m^2}{2}\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) \quad \text{(Fierz-Pauli action)}\,.$$

When $m = 0$, gauge symmetry (linearized general covariance)

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \, ,$$

$\xi_\mu(x)$: space-time dependent gauge parameter.

The combination $h_{\mu\nu}h^{\mu\nu} - h^2$:

Fierz-Pauli tuning (not related with any symmetry)

For the combination $h_{\mu\nu}h^{\mu\nu} - (1-a)h^2$,

if $a \neq 0$, there appears ghost scalar field with mass

$$m_g^2 = \frac{3-4a}{2a} m^2$$

$\left(m_g^2 \to \infty \text{ when } a \to 0\right)$.

Hamiltonian and counting of degrees of freedom:

$\frac{D(D-1)}{2} - 1$ propagating degrees of freedom in D dimensions

(5 degrees of freedom for $D = 4$).

Legendre transformation only with respect to the spatial components h_{ij}.

$$\pi_{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \dot{h}_{ij} - \dot{h}_{kk}\delta_{ij} - 2\partial_{(i}h_{j)0} + 2\partial_k h_{0k}\delta_{ij} \,,$$

$$\Rightarrow \; S = \int d^D x \left\{ \pi_{ij}\dot{h}_{ij} - \mathcal{H} + 2h_{0i}\left(\partial_j \pi_{ij}\right) + m^2 h_{0i}^2 \right.$$

$$\left. + h_{00}\left(\vec{\nabla}^2 h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii}\right) \right\} \,,$$

$$\mathcal{H} = \frac{1}{2}\pi_{ij}^2 - \frac{1}{2}\frac{1}{D-2}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik}$$

$$+ \partial_i h_{ij}\partial_j h_{kk} - \frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \frac{1}{2}m^2(h_{ij}h_{ij} - h_{ii}^2) \,.$$

$m = 0$ case: h_{0i}, h_{00}: Lagrange multipliers $\rightarrow$ constraints

$$\partial_j \pi_{ij} = 0\,, \quad \vec{\nabla}^2 h_{ii} - \partial_i \partial_j h_{ij} = 0\,.$$

First class constraints $\rightarrow$ gauge symmetry ($\Leftarrow$ general covariance)

For $D = 4$, h_{ij} and π_{ij} each have 6 components, respectively.
$\rightarrow$ 12 dimensional phase space.
4 constraints $+$ 4 gauge invariances
$\rightarrow$ 4 dimensional phase space
(two polarizations (helicities) of massless graviton)

$m \neq 0$: h_{0i} are no longer Lagrange multipliers $\delta h_{0i} \;\Rightarrow\; h_{0i} = -\frac{1}{m^2}\partial_j \pi_{ij}$,

$$S = \int d^D x \left\{ \pi_{ij}\dot{h}_{ij} - \mathcal{H} + h_{00}\left(\vec{\nabla}^2 h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii}\right)\right\} ,$$

$$\begin{aligned}\mathcal{H} =& \frac{1}{2}\pi_{ij}^2 - \frac{1}{2}\frac{1}{D-2}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} \\ & + \partial_i h_{ij}\partial_j h_{kk} - \frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \frac{1}{2}m^2\left(h_{ij}h_{ij} - h_{ii}^2\right) + \frac{1}{m^2}\left(\partial_j \pi_{ij}\right)^2 .\end{aligned}$$

h_{00}: Lagrange multiplier $\rightarrow$ single constraint

$$\mathcal{C} = -\vec{\nabla}^2 h_{ii} + \partial_i\partial_j h_{ij} + m^2 h_{ii} = 0 ,$$

Secondary constraint:

$$\{H, \mathcal{C}\}_{\mathrm{PB}} = \frac{1}{D-2}m^2\pi_{ii} + \partial_i\partial_j\pi_{ij} = 0 , \quad H = \int d^d x\, \mathcal{H} ,$$

Two second class constraints.

For $D = 4$,
12 dimensional phase space $-$ 2 constraints $= 10$ degrees of freedom
(5 polarizations of the massive graviton and their conjugate momenta).

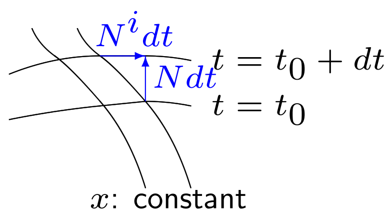
Boulware-Deser ghost

D. G. Boulware and S. Deser, "Classical General Relativity Derived from Quantum Gravity," Annals Phys. **89** (1975) 193.

In non-linear (interacting) theory, 6th degree of freedom appears as a ghost.

Non-linear massive gravity action with flat metric $\eta_{\mu\nu}$, $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$

$$S = \frac{1}{2\kappa^2}\int d^D x\left[\sqrt{-g}R - \frac{1}{4}m^2\eta^{\mu\alpha}\eta^{\nu\beta}\left(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta}\right)\right] .$$



ADM formalism (N: lapse function, N_i: shift function)

$$g_{00} = -N^2 + g^{ij}N_i N_j \ , \quad g_{0i} = N_i \ , \quad g_{ij} = g_{ij} \ .$$

$i, j, \cdots = 1, 2, 3$, g^{ij}: inverse of the spatial metric g_{ij}.

$m = 0$ case

Einstein-Hilbert action (after partial integrations)

$$\frac{1}{2\kappa^2}\int d^D x \sqrt{g} N \left[{}^{(d)}R - K^2 + K^{ij}K_{ij} \right] ,$$

${}^{(d)}R$: curvature of spatial metric g_{ij}, K_{ij}: extrinsic curvature

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right) ,$$

∇_i: covariant derivative w.r.t. the spatial metric g_{ij}.
Canonical momenta with respect to g_{ij}:

$$p^{ij} = \frac{\delta L}{\delta \dot{g}_{ij}} = \frac{1}{2\kappa^2}\sqrt{g}\left(K^{ij} - K g^{ij} \right) ,$$

Hamiltonian:

$$H = \left(\int_{\Sigma_t} d^d x \; p^{ab} \dot{g}_{ab} \right) - L = \int_{\Sigma_t} d^d x \; N \mathcal{C} + N_i \mathcal{C}^i \, .$$

$$\mathcal{C} = \sqrt{g} \left[{}^{(d)}R + K^2 - K^{ij} K_{ij} \right] , \quad \mathcal{C}^i = 2 \sqrt{g} \nabla_j \left(K^{ij} - K h^{ij} \right) ,$$

$$K_{ij} = \frac{2\kappa^2}{\sqrt{g}} \left(p_{ij} - \frac{1}{D-2} p h_{ij} \right) \, .$$

For $m = 0$, Hamiltonian vanishes. N, N_i: Lagrange multipliers
$\Rightarrow \mathcal{C} = 0$, $\mathcal{C}_i = 0$: first class constraints $\Leftrightarrow$ general covariance
In $D = 4$,
12 phase space metric components $-$ 4 constraints $-$ 4 gauge symmetries
$=$ 4 phase space degrees of freedom
$=$ degrees of freedom in linearized theory of massless spin 2 graviton

$m \neq 0$ case $(h_{ij} \equiv g_{ij} - \delta_{ij})$

$$\eta^{\mu\alpha}\eta^{\mu\beta}\left(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\mu\beta}\right)$$
$$= \delta^{ik}\delta^{jl}\left(h_{ij}h_{kl} - h_{ik}h_{jl}\right) + 2\delta^{ij}h_{ij} - 2N^2\delta^{ij}h_{ij} + 2N_i\left(g^{ij} - \delta^{ij}\right)N_i\,,$$

Action

$$S = \frac{1}{2\kappa^2}\int d^Dx \Big\{ p^{ab}\dot{g}_{ab} - N\mathcal{C} - N_i\mathcal{C}^i - \frac{m^2}{4}\left[\delta^{ik}\delta^{jl}\left(h_{ij}h_{kl} - h_{ik}h_{jl}\right) + 2\delta^{ij}h_{ij} - 2N^2\delta^{ij}h_{ij} + 2N_i\left(g^{ij} - \delta^{ij}\right)N_j\right]\Big\}\,.$$

N^2, N_iN_j terms $\Rightarrow$ N^2, N_i: *Not* Lagrange multipliers but auxiliary fields.

$$N = \frac{\mathcal{C}}{m^2 \delta^{ij} h_{ij}}\,, \qquad N_i = \frac{1}{m^2}\left(g^{ij} - \delta^{ij}\right)^{-1} \mathcal{C}^j\,.$$

No constraints nor gauge symmetries.

Hamiltonian:

$$\begin{aligned} H =& \frac{1}{2\kappa^2} \int d^d x \left\{ \frac{1}{2m^2} \frac{\mathcal{C}^2}{\delta^{ij} h_{ij}} + \frac{1}{2m^2} \mathcal{C}^i \left(g^{ij} - \delta^{ij}\right)^{-1} \mathcal{C}^j \right. \\ & \left. + \frac{m^2}{4} \left[\delta^{ik}\delta^{jl} \left(h_{ij}h_{kl} - h_{ik}h_{jl}\right) + 2\delta^{ij} h_{ij}\right] \right\}\,. \end{aligned}$$

12 phase space degrees of freedom, or 6 real degrees of freedom.
One more degree of freedom, compared with linearized theory

$\Rightarrow$ ghost scalar

Boulware-Deser ghost

vDVZ(van Dam, Veltman, and Zakharov) discontinuity

H. van Dam and M. J. G. Veltman, "Massive and massless Yang-Mills and gravitational fields," Nucl. Phys. B **22** (1970) 397.

V. I. Zakharov, "Linearized gravitation theory and the graviton mass," JETP Lett. **12** (1970) 312 [Pisma Zh. Eksp. Teor. Fiz. **12** (1970) 447].

Discontinuity of $m \to 0$ limit in the free massive gravity with the Einstein gravity due to the extra degrees of freedom in the limit.
$\Rightarrow$ the Vainstein mechanism

A. I. Vainshtein, "To the problem of nonvanishing gravitation mass," Phys. Lett. B **39** (1972) 393.

Non-linearity screens the extra degrees of freedom (non-linearity becomes strong when m is small).

Massive gravity without ghost

C. de Rham and G. Gabadadze, "Generalization of the Fierz-Pauli Action," Phys. Rev. D **82**, 044020 (2010) [arXiv:1007.0443 [hep-th]],

C. de Rham, G. Gabadadze and A. J. Tolley, "Resummation of Massive Gravity," Phys. Rev. Lett. **106** (2011) 231101 [arXiv:1011.1232 [hep-th]].

S. F. Hassan and R. A. Rosen, "Resolving the Ghost Problem in non-Linear Massive Gravity," Phys. Rev. Lett. **108** (2012) 041101 [arXiv:1106.3344 [hep-th]].

Non-dynamical metric $f_{\mu\nu}\,(\sim \eta_{\mu\nu})$, $\sqrt{g^{-1}f}$: $\sqrt{g^{-1}f}\sqrt{g^{-1}f} = g^{\mu\lambda}f_{\lambda\nu}$

Minimal extension of Fierz-Pauli action:

$$S = M_p^2 \int d^4x\sqrt{-g}\,\left[R - 2m^2\left(\mathrm{tr}\,\sqrt{g^{-1}f} - 3\right)\right]\,.$$

$\Rightarrow$ vDVZ discontinuity $\Rightarrow$

$$S = M_p^2 \int d^4x \sqrt{-g} \left[R + 2m^2 \sum_{n=0}^{3} \beta_n \, e_n(\sqrt{g^{-1}f}) \right] \,,$$

$$e_0(\mathbb{X}) = 1 \,, \quad e_1(\mathbb{X}) = [\mathbb{X}] \,, \quad e_2(\mathbb{X}) = \tfrac{1}{2}([\mathbb{X}]^2 - [\mathbb{X}^2]) \,,$$

$$e_3(\mathbb{X}) = \tfrac{1}{6}([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3]) \,,$$

$$e_4(\mathbb{X}) = \tfrac{1}{24}([\mathbb{X}]^4 - 6[\mathbb{X}]^2[\mathbb{X}^2] + 3[\mathbb{X}^2]^2 + 8[\mathbb{X}][\mathbb{X}^3] - 6[\mathbb{X}^4]) \,,$$

$$e_k(\mathbb{X}) = 0 \quad \text{for } k > 4 \,,$$

$$\mathbb{X} = (X^\mu{}_\nu) \,, \quad [\mathbb{X}] \equiv X^\mu{}_\mu \,,$$

$\sim$ Galileon $\Rightarrow$ Vainshtein mechanism

(longitudinal scalar mode ($h_{\mu\nu} \sim \partial_\mu \partial_\nu \phi$) $\sim$ Galileon scalar field)

Hamiltonian constraint: Minimal extension case

ADM formulation, $f_{\mu\nu} = \eta_{\mu\nu} \Rightarrow$

$$\mathcal{L} = \pi^{ij}\partial_t \gamma_{ij} + NR^0 + N^i R_i - 2m^2 \sqrt{\gamma}\, N \left(\mathrm{tr}\, \sqrt{g^{-1}\eta} - 3 \right) .$$

$$(g^{-1}\eta)^{\mu}{}_{\nu} = \frac{1}{N^2} \begin{pmatrix} 1 & N^l \delta_{lj} \\ -N^i & (N^2\gamma^{il} - N^i N^l)\delta_{lj} \end{pmatrix} , \quad N^i = \gamma^{ij} N_j \, .$$

Highly nonlinear action in $N_\mu \Rightarrow$ New combinations n^i

$$N^i = (\delta^i_j + N D^i{}_j) n^j \, ,$$

$$D^i{}_j : \ (\sqrt{1 - n^T \mathbf{I}\, n})\, D = \sqrt{(\gamma^{-1} - D n n^T D^T)\mathbf{I}} \, ,$$

$$\mathbf{I} = \delta_{ij} \, , \quad \mathbf{I}^{-1} = \delta^{ij} \, ,$$

$$
\begin{aligned}
\Rightarrow \ \mathcal{L} =& \pi^{ij}\partial_t\gamma_{ij} + NR^0 + R_i(\delta^i_j + N D^i{}_j)n^j \\
& - 2m^2\sqrt{\gamma}\left[\sqrt{1 - n^T\mathbf{I}n} + N\operatorname{tr}\left(\sqrt{\gamma^{-1}\mathbf{I} - Dnn^TD^T\mathbf{I}}\right) - 3N\right].
\end{aligned}
$$

Linear in N.

$\delta n_i \ \Rightarrow \ n^i = -R_j\delta^{ji}\left[4m^4\det\gamma + R_k\delta^{kl}R_l\right]^{-1/2}$: Not including N.

$$
\delta N \ \Rightarrow R^0 + R_i D^i{}_j n^j - 2m^2\sqrt{\gamma}\left[\sqrt{1 - n^r\delta_{rs}n^s}\ D^k{}_k - 3\right] = 0\,.
$$

$+$ secondary constraint $= 2$ constraints.

12 components of γ_{ij} and π^{ij} $-$ 2 constraints

$= 10$ components (massive spin 2)

Bimetric gravity (bigravity)

S. F. Hassan and R. A. Rosen, "Bimetric Gravity from Ghost-free Massive Gravity," JHEP **1202** (2012) 126 [arXiv:1109.3515 [hep-th]].

Dynamical $f_{\mu\nu}$ (background independent).

$$
\begin{aligned}
S =& M_g^2 \int d^4x \sqrt{-\det g}\, R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f}\, R^{(f)} \\
&+ 2m^2 M_{\text{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n\, e_n(\sqrt{g^{-1}f})\,, \\
& 1/M_{\text{eff}}^2 \equiv 1/M_g^2 + 1/M_f^2\,.
\end{aligned}
$$

$R^{(g)}$: scalar curvature for $g_{\mu\nu}$, $R^{(f)}$: scalar curvature for $f_{\mu\nu}$.

Spectrum of the linearized theory

Minimal case: $\beta_0 = 3$, $\beta_1 = -1$, $\beta_2 = 0$, $\beta_3 = 0$, $\quad \beta_4 = 1$.

Linearize $\qquad g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_g} h_{\mu\nu}\,, \quad f_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_f} l_{\mu\nu}\,,$

$$\Rightarrow \; S = \int d^4x \, (h_{\mu\nu} \hat{\mathcal{E}}^{\mu\nu\alpha\beta} h_{\alpha\beta} + l_{\mu\nu} \hat{\mathcal{E}}^{\mu\nu\alpha\beta} l_{\alpha\beta})$$

$$- \frac{m^2 M_{\text{eff}}^2}{4} \int d^4x \left[\left(\frac{h^{\mu}{}_{\nu}}{M_g} - \frac{l^{\mu}{}_{\nu}}{M_f} \right)^2 - \left(\frac{h^{\mu}{}_{\mu}}{M_g} - \frac{l^{\mu}{}_{\mu}}{M_f} \right)^2 \right] .$$

$\hat{\mathcal{E}}^{\mu\nu\alpha\beta}$: usual Einstein-Hilbert kinetic operator.

Change of variables

$$\frac{1}{M_{\rm eff}}\, u_{\mu\nu} = \frac{1}{M_f}\, h_{\mu\nu} + \frac{1}{M_g}\, l_{\mu\nu}\,, \qquad \frac{1}{M_{\rm eff}}\, v_{\mu\nu} = \frac{1}{M_g}\, h_{\mu\nu} - \frac{1}{M_f}\, l_{\mu\nu}\,.$$

$\Rightarrow$

$$S = \int d^4x \, (u_{\mu\nu} \hat{\mathcal{E}}^{\mu\nu\alpha\beta} u_{\alpha\beta} + v_{\mu\nu} \hat{\mathcal{E}}^{\mu\nu\alpha\beta} v_{\alpha\beta})$$
$$- \frac{m^2}{4} \int d^4x \, \left(v^{\mu\nu} v_{\mu\nu} - v^{\mu}{}_{\mu} v^{\nu}{}_{\nu} \right) \, .$$

One massless spin-2 particle $u_{\mu\nu}$ and one massive spin-2 particle $v_{\mu\nu}$ with mass m.

$F(R)$ bigravity

Standard $F(R)$ gravity $\Leftrightarrow$ scalar tensor theory

$$S_{F(R)} = \int d^4x \sqrt{-g} \left(\frac{F(R)}{2\kappa^2} + \mathcal{L}_{\rm matter} \right) .$$

Introducing the auxiliary field A,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ F'(A) \left(R - A \right) + F(A) \right\} .$$

Variation of $A \Rightarrow A = R$: original action

Rescaling of metric

$$g_{\mu\nu} \to \mathrm{e}^{\sigma} g_{\mu\nu}\,, \quad \sigma = -\ln F'(A)\,.$$

$\Rightarrow$ Einstein frame action (Einstein-Hilbert action + real scalar field σ):

$$S_E = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R - \frac{3}{2}g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma - V(\sigma)\right)\,,$$

$$V(\sigma) = \mathrm{e}^{\sigma} g\left(\mathrm{e}^{-\sigma}\right) - \mathrm{e}^{2\sigma} f\left(g\left(\mathrm{e}^{-\sigma}\right)\right) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2}\,.$$

$A = g\left(\mathrm{e}^{-\sigma}\right) \Leftarrow \sigma = -\ln\left(1 + f'(A)\right) = -\ln F'(A)$

Coupling of σ with matters appears by the rescaling $g_{\mu\nu} \to \mathrm{e}^{\sigma} g_{\mu\nu}$.

Construction of $F(R)$ bigravity
(by the inverse process from scalar-tensor form)

Adding the following actions to the bigravity action

$$
\begin{aligned}
S_\varphi = & - M_g^2 \int d^4x \sqrt{-\det g} \left\{ \frac{3}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi) \right\} \\
& + \int d^4x \mathcal{L}_{\mathrm{matter}} \left(\mathrm{e}^{\varphi} g_{\mu\nu}, \Phi_i \right) \ , \\
S_\xi = & - M_f^2 \int d^4x \sqrt{-\det f} \left\{ \frac{3}{2} f^{\mu\nu} \partial_\mu \xi \partial_\nu \xi + U(\xi) \right\} \ .
\end{aligned}
$$

Scale transformations $g_{\mu\nu} \to \mathrm{e}^{-\varphi} g_{\mu\nu}$, $f_{\mu\nu} \to \mathrm{e}^{-\xi} f_{\mu\nu}$,

$$\begin{aligned} S_F =& M_f^2 \int d^4x \sqrt{-\det f^{\mathrm{J}}} \left\{ \mathrm{e}^{-\xi} R^{\mathrm{J}(f)} - \mathrm{e}^{-2\xi} U(\xi) \right\} \\ & + 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}} \sum_{n=0}^{4} \beta_n \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) \\ & + M_g^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}} \left\{ \mathrm{e}^{-\varphi} R^{\mathrm{J}(g)} - \mathrm{e}^{-2\varphi} V(\varphi) \right\} \\ & + \int d^4x \mathcal{L}_{\mathrm{matter}} \left(g_{\mu\nu}^{\mathrm{J}}, \Phi_i \right) . \end{aligned}$$

Kinetic terms of φ and ξ vanish. ($\mathcal{O}^J$: quantities in the Jordan fame)

Coupling of φ with matters also disappears.

Variation of φ and ξ $\Rightarrow$

$$
\begin{aligned}
0 =& 2m^2 M_{\mathrm{eff}}^2 \sum_{n=0}^{4} \beta_n \left(\frac{n}{2} - 2\right) \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right) \\
& + M_g^2 \left\{ -\mathrm{e}^{-\varphi} R^{\mathrm{J}(g)} + 2\mathrm{e}^{-2\varphi} V(\varphi) + \mathrm{e}^{-2\varphi} V'(\varphi) \right\} , \\
0 =& -2m^2 M_{\mathrm{eff}}^2 \sum_{n=0}^{4} \frac{\beta_n n}{2} \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right) \\
& + M_f^2 \left\{ -\mathrm{e}^{-\xi} R^{\mathrm{J}(f)} + 2\mathrm{e}^{-2\xi} U(\xi) + \mathrm{e}^{-2\xi} U'(\xi) \right\} .
\end{aligned}
$$

In principle, can be solved algebraically with respect to φ and ξ

$$
\varphi = \varphi\left(R^{(g)}, R^{(f)}, e_n\left(\sqrt{g^{-1} f}\right)\right) , \quad \xi = \xi\left(R^{(g)}, R^{(f)}, e_n\left(\sqrt{g^{-1} f}\right)\right) .
$$

$\Rightarrow$ analogue of $F(R)$ gravity:

$$S_F = M_f^2 \int d^4x \sqrt{-\det f^{\mathrm{J}}} F^{(f)} \left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n \left(\sqrt{g^{\mathrm{J}-1} f^{\mathrm{J}}} \right) \right)$$

$$+ 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi\left(R^{\mathrm{J}(g)}, e_n\left(\sqrt{g^{\mathrm{J}-1} f^{\mathrm{J}}}\right)\right)} e_n \left(\sqrt{g^{\mathrm{J}-1} f^{\mathrm{J}}} \right)$$

$$+ M_g^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}} F^{\mathrm{J}(g)} \left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n \left(\sqrt{g^{\mathrm{J}-1} f^{\mathrm{J}}} \right) \right)$$

$$+ \int d^4x \mathcal{L}_{\mathrm{matter}} \left(g^{\mathrm{J}}_{\mu\nu}, \Phi_i \right) \ ,$$

Here

$$F^{\mathrm{J}(g)}\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}^{-1}}f^{\mathrm{J}}}\right)\right) \equiv \left\{\mathrm{e}^{-\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}^{-1}}f^{\mathrm{J}}}\right)\right)}R^{\mathrm{J}(g)}\right.$$

$$\left.-\mathrm{e}^{-2\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}^{-1}}f^{\mathrm{J}}}\right)\right)}V\left(\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}^{-1}}f^{\mathrm{J}}}\right)\right)\right)\right\},$$

$$F^{(f)}\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}^{-1}}f^{\mathrm{J}}}\right)\right) \equiv \left\{\mathrm{e}^{-\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}^{-1}}f^{\mathrm{J}}}\right)\right)}R^{\mathrm{J}(f)}\right.$$

$$\left.-\mathrm{e}^{-2\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}^{-1}}f^{\mathrm{J}}}\right)\right)}U\left(\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}^{-1}}f^{\mathrm{J}}}\right)\right)\right)\right\}.$$

It is difficult to explicitly solve equations with respect to φ and ξ and it might be better to define the model by introducing the auxiliary scalar fields φ and ξ.

Cosmological Reconstruction

Usually we start from a given model and investigate the development of the universe etc. by using the given equations. Here we consider the inverse, that is, for a given development of the universe, we construct a model which reproduces the development, which we call reconstruction.

Minimal case:

$$\begin{aligned} S_{\mathrm{bi}} =& M_g^2 \int d^4x \sqrt{-\det g}\, R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f}\, R^{(f)} \\ & + 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \left(3 - \mathrm{tr}\, \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f} \right) \,. \end{aligned}$$

Start from the Einstein frame action. Neglect matter.

$\delta g_{\mu\nu} \;\Rightarrow$

$$
\begin{aligned}
0 &= M_g^2 \left(\frac{1}{2} g_{\mu\nu} R^{(g)} - R^{(g)}_{\mu\nu}\right) \\
&+ m^2 M_{\text{eff}}^2 \left\{ g_{\mu\nu}\left(3 - \operatorname{tr}\sqrt{g^{-1}f}\right) + \frac{1}{2} f_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}{}_{\nu} + \frac{1}{2} f_{\nu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}{}_{\mu} \right\} \\
&+ M_g^2 \left[\frac{1}{2}\left(\frac{3}{2} g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi + V(\varphi)\right) g_{\mu\nu} - \frac{3}{2}\partial_\mu\varphi\partial_\nu\varphi\right] .
\end{aligned}
$$

$\delta f_{\mu\nu} \;\Rightarrow$

$$
\begin{aligned}
0 &= M_f^2 \left(\frac{1}{2} f_{\mu\nu} R^{(f)} - R^{(f)}_{\mu\nu}\right) \\
&+ m^2 M_{\text{eff}}^2 \sqrt{\det\left(f^{-1}g\right)} \left\{ -\frac{1}{2} f_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{\rho}{}_{\nu} - \frac{1}{2} f_{\nu\rho}\left(\sqrt{g^{-1}f}\right)^{\rho}{}_{\mu} + \det\left(\sqrt{g^{-1}f}\right) f_{\mu\nu} \right\} \\
&+ M_f^2 \left[\frac{1}{2}\left(\frac{3}{2} f^{\rho\sigma}\partial_\rho\xi\partial_\sigma\xi + U(\xi)\right) f_{\mu\nu} - \frac{3}{2}\partial_\mu\xi\partial_\nu\xi\right] .
\end{aligned}
$$

$\delta\varphi$, $\delta\xi$ $\Rightarrow$

$$0 = -3\Box_g \varphi + V'(\varphi)\,, \quad 0 = -3\Box_f \xi + U'(\xi)\,.$$

$\Box_g$, $\Box_f$: d'Alembertian w.r.t. g, f.

Bianchi identity $0 = \nabla_g^\mu \left(\frac{1}{2} g_{\mu\nu} R^{(g)} - R^{(g)}_{\mu\nu}\right)$ + field equations $\Rightarrow$

$$0 = - g_{\mu\nu} \nabla_g^\mu \left(\mathrm{tr} \sqrt{g^{-1} f} \right) + \frac{1}{2} \nabla_g^\mu \left\{ f_{\mu\rho} \left(\sqrt{g^{-1} f} \right)^{-1\,\rho}_{\quad\;\nu} + f_{\nu\rho} \left(\sqrt{g^{-1} f} \right)^{-1\,\rho}_{\quad\;\mu} \right\}\,.$$

Similarly

$$0 = \nabla_f^\mu \left[\sqrt{\det\left(f^{-1} g\right)} \left\{ -\frac{1}{2} \left(\sqrt{g^{-1} f} \right)^{-1\nu}_{\quad\;\sigma} g^{\sigma\mu} - \frac{1}{2} \left(\sqrt{g^{-1} f} \right)^{-1\mu}_{\quad\;\sigma} g^{\sigma\nu} + \det\left(\sqrt{g^{-1} f} \right) f^{\mu\nu} \right\} \right]\,.$$

In case of the Einstein gravity,
conservation law $\Leftarrow$ Einstein equation $+$ Bianchi identity
or conservation laws $\Leftarrow$ scalar field equations

Scalar field equations are not independent of Einstein equation.

In case of bigravity,
only conservation laws $\Leftarrow$ scalar field equations
Einstein equation $+$ Bianchi identities $+$ scalar field equations
$\Rightarrow$ new equations independent of Einstein equation.

Scalar field equations are independent of Einstein equation.

Assume FRW universes by using the conformal time t

$$ds_g^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu = a(t)^2 \left(-dt^2 + \sum_{i=1}^{3} \left(dx^i \right)^2 \right) ,$$

$$ds_f^2 = \sum_{\mu,\nu=0}^{3} f_{\mu\nu} dx^\mu dx^\nu = -c(t)^2 dt^2 + b(t)^2 \sum_{i=1}^{3} \left(dx^i \right)^2 .$$

(Most general form assuming the homogeneity, isometry, and flat spacial part.)

$\Rightarrow$

$$\begin{aligned} \delta g_{tt} \ : \ 0 =& -3M_g^2 H^2 - 3m^2 M_{\mathrm{eff}}^2 \left(a^2 - ab \right) + \left(\frac{3}{4} \dot{\varphi}^2 + \frac{1}{2} V(\varphi) a(t)^2 \right) M_g^2 , \\ \delta g_{ij} \ : \ 0 =& M_g^2 \left(2\dot{H} + H^2 \right) + m^2 M_{\mathrm{eff}}^2 \left(3a^2 - 2ab - ac \right) \\ & + \left(\frac{3}{4} \dot{\varphi}^2 - \frac{1}{2} V(\varphi) a(t)^2 \right) M_g^2 , \qquad H \equiv \frac{\dot{a}}{a} . \end{aligned}$$

H is not exactly Hubble rate $\Leftarrow$ t: conformal time

$$\delta f_{tt} \; : \; 0 = -3M_f^2 K^2 + m^2 M_{\mathrm{eff}}^2 c^2 \left(1 - \frac{a^3}{b^3}\right) + \left(\frac{3}{4}\dot\xi^2 - \frac{1}{2}U(\xi)c(t)^2\right) M_f^2 \,,$$

$$\begin{aligned} \delta f_{ij} \; : \; 0 =& M_f^2 \left(2\dot K + 3K^2 - 2LK\right) + m^2 M_{\mathrm{eff}}^2 \left(\frac{a^3 c}{b^2} - c^2\right) \\ &+ \left(\frac{3}{4}\dot\xi^2 - \frac{1}{2}U(\xi)c(t)^2\right) M_f^2 \,. \\ & K \equiv \dot b/b \,, \quad L = \dot c/c \,. \end{aligned}$$

Both of equations derived from Bianchi identity:

$cH = bK$ or $\frac{c\dot a}{a} = \dot b$.

If $\dot a \neq 0$, we obtain $c = a\dot b/\dot a$.
If $\dot a = 0$, we find $\dot b = 0$, that is, a, b: constant, c can be arbitrary.

Redefinition of the scalar fields: $\varphi = \varphi(\eta)$, $\xi = \xi(\zeta)$.

Identify $\eta = \zeta = t \Rightarrow$

$$\begin{aligned}
\omega(t) M_g^2 =& -4M_g^2\left(\dot{H} - H^2\right) - 2m^2 M_{\rm eff}^2 (ab - ac)\,,\\
\tilde{V}(t) a(t)^2 M_g^2 =& M_g^2\left(2\dot{H} + 4H^2\right) + m^2 M_{\rm eff}^2 (6a^2 - 5ab - ac)\,,\\
\sigma(t) M_f^2 =& -4M_f^2\left(\dot{K} - LK\right) - 2m^2 M_{\rm eff}^2\left(-\frac{c}{b} + 1\right)\frac{a^3 c}{b^2}\,,\\
\tilde{U}(t) c(t)^2 M_f^2 =& M_f^2\left(2\dot{K} + 6K^2 - 2LK\right) + m^2 M_{\rm eff}^2\left(\frac{a^3 c}{b^2} - 2c^2 + \frac{a^3 c^2}{b^3}\right)\,.
\end{aligned}$$

$$\omega(\eta) = 3\varphi'(\eta)^2,\ \tilde{V}(\eta) = V\left(\varphi\left(\eta\right)\right),\ \sigma(\zeta) = 3\xi'(\zeta)^2,\ \tilde{U}(\zeta) = U\left(\xi\left(\zeta\right)\right)\,.$$

For arbitrary $a(t)$ and $b(t)$, if we choose $\omega(t)$, $\tilde{V}(t)$, $\sigma(t)$, and $\tilde{U}(t)$ to satisfy the above equations, a model admitting the given $a(t)$ and $b(t)$ evolution can be reconstructed.

Cosmological Models

FRW universe:: $ds^2 = \tilde{a}(t)^2 \left(-dt^2 + \sum_{i=1}^{3} \left(dx^i\right)^2\right)$.

$\tilde{a}(t)^2 = \frac{l^2}{t^2}$: de Sitter universe.

$\tilde{a}(t)^2 = \frac{l^{2n}}{t^{2n}}$ with $n \neq 1$ case:

Redefinition of time coordinate: $d\tilde{t} = \pm\frac{l^n}{t^n}dt \left(\tilde{t} = \pm\frac{l^n}{n-1}t^{1-n}\right)$

$$\Rightarrow \ ds^2 = -d\tilde{t}^2 + \left(\pm(n-1)\frac{\tilde{t}}{l}\right)^{-\frac{2n}{1-n}} \sum_{i=1}^{3} \left(dx^i\right)^2 .$$

$0 < n < 1$: phantom universe, $n > 1$: quintessence universe, $n < 0$: decelerating universe

Universe with $a(t) = b(t) = 1$

$a(t) = b(t) = 1$ satisfies the previous constraint.
$\Rightarrow$ Einstein frame metric $g_{\mu\nu}$: flat Minkowski space
Physical metric: the scalar field does not directly coupled with matter.
$\Rightarrow$ the metric we observe: Jordan frame metric $g^{\mathrm{J}}_{\mu\nu} = \mathrm{e}^{\varphi} g_{\mu\nu}$.

$$\begin{aligned}
\omega(t) M_g^2 =& 12 M_g^2 \tilde{H}^2 = 2m^2 M_{\mathrm{eff}}^2 (c-1) \ , \\
\tilde{V}(t) M_g^2 =& m^2 M_{\mathrm{eff}}^2 (1-c) = -6 M_g^2 \tilde{H}^2 \ \Rightarrow \ c = 1 + \frac{6\tilde{H}^2 M_g^2}{m^2 M_{\mathrm{eff}}^2} \, , \\
\sigma(t) M_f^2 =& 2m^2 M_{\mathrm{eff}}^2 (c-1) = 12 M_g^2 \tilde{H}^2 \, , \\
\tilde{U}(t) M_f^2 =& m^2 M_{\mathrm{eff}}^2 c (1-c) = -6 M_g^2 \tilde{H}^2 \left(1 + \frac{6\tilde{H}^2}{m^2 M_{\mathrm{eff}}^2} \right) \ .
\end{aligned}$$

Note: $\omega(t)$, $\sigma(t) > 0$ (no ghost)

Big Rip, quintessence, de Sitter and decelerating universes

$$\tilde{a}(t)^2 = \frac{l^{2n}}{t^{2n}}$$

$$\omega(\eta)^2 M_g^2 = \frac{12n^2 M_g^2}{\eta^2}\,, \quad \tilde{V}(\eta) M_g^2 = -\frac{6n^2 M_g^2}{\eta^2}\,,$$

$$\sigma(\zeta) M_f^2 = \frac{12n^2 M_g^2}{\zeta^2}\,, \quad \tilde{U}(\zeta) M_f^2 = -\frac{6n^2 M_g^2}{\zeta^2}\left(1 + \frac{6n^2}{m^2 M_{\mathrm{eff}}^2 \zeta^2}\right)\,.$$

$$\Rightarrow \quad \mathrm{e}^{\xi} = \frac{n^2}{t^2}\,,$$

$$\left(ds_f^{\mathrm{J}}\right)^2 = \sum_{\mu,\nu=0}^{3} f_{\mu\nu}^{\mathrm{J}} dx^\mu dx^\nu = \mathrm{e}^{\xi} ds_f^2 = \frac{n^2}{t^2}\left\{ -\left(1 + \frac{6n^2}{m^2 M_{\mathrm{eff}}^2 t^2}\right)^2 dt^2 + \left(dx^i\right)^2 \right\}\,.$$

When $t \sim 0$, redefinition:

$$\tilde{t} \sim \frac{\alpha}{2t^2}\,, \quad \alpha \equiv \frac{6n^3}{m^2 M_{\mathrm{eff}}^2 t^2}\,,$$

$\Rightarrow$

$$\left(ds_f^{\mathrm{J}}\right)^2 \sim -d\tilde{t}^2 + \frac{2n^2\tilde{t}}{\alpha}\left(dx^i\right)^2 \,.$$

$t \to 0$ (Big Bang or Big Rip) $\Leftrightarrow \tilde{t} \to +\infty$.

There does not occur singularity in the metric $\left(ds_f^{\mathrm{J}}\right)^2$ because the scale factor $\tilde{a}$ which is proportional to $\tilde{t}$ corresponds to the universe filled with radiation.

Super-luminal mode in bigravity

There can be a signal whose speed is larger than the speed of light.
Speed v_g of the massless particle which propagates in the universe described by $g^{\mathrm{J}}_{\mu\nu}$ or $g_{\mu\nu}$

$$v_g^2 = (dx/dt)^2 = 1 \ \Leftrightarrow \ \text{special relativity.}$$

Speed v_f in $f^{\mathrm{J}}_{\mu\nu}$ or $f_{\mu\nu}$

$$v_f^2 = (dx/dt)^2 = c(t)^2/b(t)^2$$

If $c(t)/b(t) > 1$, $v_f > 1$ speed of light in g universe.
$c(t) > 1$ except of $\tilde{H} = 0$: $v_f = 1 + \frac{6\tilde{H}^2}{m^2 M_{\mathrm{eff}}^2} > 1$.
v_f is greater than the speed of light. (Causality is not always violated.)

Summary

- $F(R)$ bigravity in the conventional description with two metrics g and f.
- Explicit and exact solution of FRW equations, Big (and Little) Rip, de Sitter, quintessence and decelerating universes.
- In general, the physical g cosmological singularity is manifested as metric f cosmological singularity. However, there are examples where cosmological singularity of physical g universe does not occur in the universe described by reference metric f and vice-versa.
- The massless particle in the space-time given by the metric $f_{\mu\nu}$ or $f^{\mathrm{J}}_{\mu\nu}$ can be super-luminal.
- Other models, scalar-tensor, Brans-Dicke.

Bounce Cosmology

Matter bounce scenario

1. In the initial phase of the contraction, the universe is at the matter-dominated stage.

2. There happens a bounce without any singularity.

3. The primordial curvature perturbations with the observed spectrum can also be generated.

K. Bamba, A. N. Makarenko, A. N. Myagky, S. Nojiri and S. D. Odintsov,
"Bounce cosmology from $F(R)$ gravity and $F(R)$ bigravity,"
JCAP01(2014)008 [arXiv:1309.3748 [hep-th]].

Standard $F(R)$ gravity

$$a \sim \mathrm{e}^{\alpha t^2} \;\Leftrightarrow\; F(R) = \frac{1}{\alpha} R^2 - 72R + 144\alpha\,.$$

This kind of model also realizes the Starobinsky inflation.

$F(R)$ bigravity

$$\omega(\eta) = 12\alpha M_g^2 \eta^2\,, \quad \tilde{V}(\eta) = -12\alpha^2\eta^2\,,$$
$$\sigma(\zeta) = \frac{24\alpha^2 M_g^2 \zeta^2}{M_f^2}\,, \quad \tilde{U}(\zeta) = -\frac{12\alpha^2 M_g^2 \zeta^2}{M_f^2}\left(1 + \frac{12\alpha^2 M_g^2 \zeta^2}{m^2 M_{\mathrm{eff}}^2}\right)\,.$$